REAL ANALYSIS PRELIMINARY EXAM
JANUARY 10, 2026

All responses require justification.

(1) (a) Prove that for all $A, B > 0$, we have $A^{\ln B} = B^{\ln A}$.

(b) Prove that $\displaystyle\sum_{n=2}^{\infty} \frac{1}{(\ln n)^{\ln n}}$ converges.

(2) Let K be a compact subset of $\mathbb{R}$ and S be a closed subset of $\mathbb{R}$ such that $K \cap S = \emptyset$. Prove that there exist disjoint open sets U and V such that $K \subset U$ and $S \subset V$.

(3) Let f and g be continuous on $[a, b]$ with $g > 0$ on $[a, b]$. Prove that there exists $c \in [a, b]$ such that $\displaystyle\int_a^b f(x)g(x)\,dx = f(c)\int_a^b g(x)\,dx$. *(You may not simply quote a theorem; give a proof.)*

(4) Consider the series $f(x) = \displaystyle\sum_{n=1}^{\infty} \frac{1}{1 + x \cdot n^2}$ for $x > 0$.

(a) Prove that f defines a continuous function on $(0, \infty)$.

(b) Find $\displaystyle\lim_{x\to\infty} f(x)$.

(c) Find $\displaystyle\lim_{x\to 0^+} f(x)$.

(5) Let $f : (0, \infty) \to \mathbb{R}$ be a twice continuously differentiable function. Suppose that $f''(x)$ has exactly one zero, and suppose that $f(x) \to 0$ as $x \to \infty$. Prove that $f'(x) \to 0$ as $x \to \infty$.

(6) Let Ω be a bounded region in $\mathbb{R}^3$ whose boundary $\partial\Omega$ is a smooth, oriented surface.

(a) Use the divergence theorem to express the volume of Ω as the integral of a two-form over $\partial\Omega$.

(b) *Using part (a) and modified spherical coordinates,* compute the volume of the region bounded by the ellipsoid $x^2 + y^2 + 4z^2 = 4$.

(7) Let $f : \mathbb{R} \to \mathbb{R}$ be the function of period 2π such that $f(x) = |x|$ for $-\pi \le x \le \pi$.

(a) Find the Fourier series for f.

(b) At which points $x \in \mathbb{R}$ does the Fourier series converge to $f(x)$?

(c) Use your answer to find the value of $\displaystyle\sum_{n=1}^{\infty} \frac{1}{(2n-1)^2}$.

(8) Let $\varphi(x, y, z) = y^2 z + x + 2x^4$. Suppose there exists a differentiable function $f(y, z)$ defined on an open set $U \subset \mathbb{R}^2$ such that $\varphi(f(y, z), y, z) = 1$ for all $(y, z) \in U$.

(a) Find the set of all (y, z) that cannot possibly be in U.

(b) Compute $\dfrac{\partial f}{\partial y}$ and $\dfrac{\partial f}{\partial z}$ as functions of $(y, z) \in U$.